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Solitons and Confinement*

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Abstract

The Thirring and the Schwinger models both massless and massive are discussed as prototypes for theories with topological quantum numbers and confinement respectively.
Bosonization of the fermion field is introduced from the beginning allowing a unified treatment of various models. An analysis of their charge sectors clarifies the relation between the periodicity of the potential in the bosonized version of those models and the existence of an additive quantum number.
A brief outline of the essential features discussed which may survive in 4-dimensional space-time is made in the last section.

Contents

1. Introduction

This paper will be devoted to a discussion of some aspects of two fascinating and in a way complementary recent developments in quantum field theory.
The first one has to do with the fact that some systems have conservation laws totally unrelated to any Noether Symmetry exibited by their Lagrangean. These conservation laws might lead to new quantum numbers, whose topological origin has been extensively presented in [*1*], and which can be associated with "extended particles" such as the soliton [*2*], the kink [*3*] and the 't Hooft-Polyakov monopole [*4*].
On the other hand it might happen that a quantum number one would read off from a symmetry of the Lagrangean will not exist in the physical state space (charge-screening) and even more, that the particles one would suppose are carriers of that quantum num-

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ber are absent from the physical spectrum of states (confinement). Such a possibility is realized in the Schwinger model [5, 6] and is expected to hold in some non-abelian gauge theories [7] where it could provide a most natural explanation for the non-observability of quarks.

The major part of this article will review two-dimensional models. Two dimensional space-time despite all its peculiarities has proved many times to be a fruitfull theoretical laboratory where one can test a number of ideas in soluble models and many times draw inspiration for more realistic theories. Needless to say, great care should be exercised in order to separate those features which are unique in two dimensions from thos which have a chance of surviving a dimensional boost. Surprisingly, in recent years, many aspects of two-dimensional models once regarded as dimensional pathologies have found close analogies in four dimensional space-time.

In section 2 we will start exploiting the possibility of parametrizing a two-dimensional fermion field in terms of boson variables (bosonization). Bosonization has its historical roots in [8]. Explicit bosonization formulas where used in [6] for the Schwinger model and in [9] for the Thirring model, and found a remarkable application in Coleman's equivalence proof [2] between the massive Thirring and since-Gordon theories.

The sine-Gordon theory is a prototype for theories with topological quantum numbers [10] and the one which is most completely understood [18, 22].

In section 3 the Schwinger model [5, 6] and the massive Schwinger model [11] will be discussed, as prototypes for field theoretical confinement.

In section 4 we will attempt a unified discussion of the models presented in the preceding sections from the point of view of charge sectors.

In section 5 a brief outline of 4-dimensional analogs of the essential features of the models previously presented will be made.

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2. The Thirring Model

a) Fermions in Terms of Bosons

As a preliminary step in the "bosonization" of the Thirring model [12] let us recall some properties of the scalar massless field in two dimension. It is well known [13] that the quantization of this field requires a "Hilbert space" with indefinite metric. This follows at once from the fact that the formal two point function of the theory has an infrared divergence ($\sim \int dp_1/|p_1|$), and by defining the correct two point function as the finite part of the divergent integral one loses its formal positivity property. One can therefore introduce a massless free field in two dimensions through the two point function

$$\langle 0'|\, \phi(x)\, \phi(y)\, |0'\rangle = -\frac{1}{4\pi} \log \{[-(x-y)^2 + i\varepsilon(x_0 - y_0)]\, \mu^2\} \tag{2.1}$$

and the requirement that all higher truncated functions vanish. The prime on the vacuum serves as reminder that by reconstructing the theory from the Wightman functions [13] one will be led to a space with indefinite metric. The mass μ^2 is an arbitrary regulator mass.

As in any free field theory one can introduce composite fields corresponding to Wick ordered polynomials or even more general functions of the free field. Of particular interest to us will be the Wick ordered exponentials of the free field,

$$:\exp\{i\lambda\phi(x)\}: = \exp\{i\lambda\phi^-(x)\}\exp\{i\lambda\phi^+(x)\} \tag{2.2}$$

with ϕ^-, ϕ^+ the creation resp. anihilation parts of the free field. Those exponentials form the main building blocks of all the soluble two dimensional models and are the basis for the bosonization formulas. The peculiar feature which allows the use of such exponentials in interesting models is the fact that they can be inbedded in a positive metric Hilbert space [*13, 14*]. To see that consider first a general Wightman function for exponentials of a massive free field, which live in a positive metric Hilbert space

$$\langle 0|\ :\exp\{i\lambda_1\phi(x_1)\}:\dots:\exp\{i\lambda_n\phi(x_n)\}:\ |0\rangle_m =$$
$$= \exp\{\sum_{i<j} - \lambda_i\lambda_j\ \Delta^+(x_i - x_j, m)\}. \tag{2.3}$$

Since for vanishing mass the two point function of the massive field behaves as

$$\Delta^+(x-y, m) = \langle 0|\ \phi(x)\ \phi(y)\ |0\rangle_m$$
$$= -\frac{1}{4\pi}\log\left[-\{(x-y)^2 - i\varepsilon(x_0 - y_0)\}\frac{m^2e^{2\gamma}}{4}\right] + O(m^2) \tag{2.4}$$

where γ is Euler's constant and the r.h.s. of (2.3) can be rewritten as

$$\exp\{\sum_{i<j} - \lambda_i\lambda_j\,\Delta^+(x_i - x_j, m)\} = \exp\left\{\sum_{i<j} - \lambda_i\lambda_j\left\{\Delta^+(x_i - x_j, m) + \frac{1}{4\pi}\log\left(\frac{m^2e^{2\gamma}}{4\mu^2}\right)\right\}\right\}$$
$$\times\exp\left[-\sum_i\frac{\lambda_i^2}{8\pi}\log\left(\frac{m^2e^{2\gamma}}{4\mu^2}\right)\right]\exp\left[\frac{+1}{8\pi}(\Sigma\lambda_i)^2\log\left(\frac{m^2e^{2\gamma}}{4\mu^2}\right)\right] \tag{2.5}$$

we readily find

$$\lim_{m\to 0}\langle 0|\left(\frac{m^2e^{2\gamma}}{4\pi^2}\right)^{\lambda_1^2/8\pi}:\exp i\lambda_1\phi(x):\dots\left(\frac{m^2e^{2\gamma}}{4\mu^2}\right)^{\lambda_n^2/8\pi}:\exp i\lambda_n\phi(x_n):\ |0\rangle_m$$
$$= \delta_{\Sigma\lambda_i,0}\langle 0'|\ :\exp i\lambda_1\phi(x):\dots:\exp i\lambda_n\phi(x_n):\ |0'\rangle. \tag{2.6}$$

Defining new Wightman functions

$$\langle 0|\ e_{\lambda_1}(x_1)\dots e_{\lambda_n}(x_n)\ |0\rangle = \delta_{\Sigma\lambda_i,0}\langle 0'|\ :\exp i\lambda_1\phi(x_1):\dots:\exp i\lambda_n\phi(x_n):\ |0'\rangle \tag{2.7}$$

which belong to a positive metric Hilbert space since they are limits of functions satisfying positivity we can identify

$$e_\lambda(x) = :\exp i\lambda\phi(x): \tag{2.8}$$

provided we associate to the exponential a conserved charge λ. In the following this selection rule which allows the inbedding in a positive metric space will be always understood whenever exponentials of zero mass fields appear.

From the well known fact that the zero mass field can be written as

$$\phi(x) = \phi(u) + \phi(v)$$
$$u = x_0 + x_1, \qquad v = x_0 - x_1 \tag{2.9}$$

where $\phi(u)$ and $\phi(v)$ are independent fields (the reader should not be confused by our notational simplification of calling all fields ϕ) the Wightman functions (2.7) factors into a product of a function of the u's and function of the v's, so we can introduce a u Hilbert space and a v Hilbert space. This allows one to enlarge the class of operators that can live in a positive metric space, since the Wightman functions defined by

$$\langle 0|\, e_{\alpha_1\delta_1}(x_1) \ldots e_{\alpha_n\alpha_n}(x_n)\, |0\rangle$$

$$= \delta_{\Sigma\alpha_{i},0}\, \delta_{\Sigma\beta_{i},0} \langle 0'|\, :\exp i\big(\alpha_1\phi(u_1) + \delta_1\phi(v_1)\big): \ldots :\exp i\big(\alpha_n\phi(u_n) + \alpha_n\phi(v_n)\big):\, |0'\rangle \tag{2.10}$$

satisfy the positivity property. Again we identify

$$e_{\alpha\delta}(x) = :\exp i\big(\alpha\phi(u) + \delta\phi(v)\big): \tag{2.11}$$

by assigning to the exponential a conserved charge α and a conserved charge δ.
Finally let us introduce the Thirring field [9] in bosonic language:

$$\begin{aligned} \psi_1 &= \left(\frac{\mu}{2\pi}\right)^{1/2} :\exp i\big(\alpha\phi(u) + \delta\phi(v)\big): \\ \psi_2 &= \left(\frac{\mu}{2\pi}\right)^{1/2} :\exp\Big(-i\big(\delta\phi(u) + \alpha\phi(v)\big)\Big): \end{aligned} \tag{2.12}$$

in the basis where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.13}$$

The parameter μ ist the regulator mass of (2.1) which in the following we set equal to one.
In order to reproduce the usual commutation relations [12] between the upper and lower components of ψ an additional Klein transformation has to be introduced. We need not worry about that for the moment and only remark here that such a transformation is possible since, due to the selection rule built in the exponentials we will have charge and pseudo charge conservation.
Consider now the Dirac operator acting on ψ

$$\begin{aligned} i\,\partial_u\psi_2 &= \frac{\delta}{\sqrt{2\pi}} :\exp -i\big(\delta\phi(u) + \alpha\phi(v)\,\partial_u\phi: = \delta :\partial_u\phi\psi_2: \\ i\,\partial_v\psi_1 &= -\frac{\delta}{\sqrt{2\pi}} :\exp i\big(\alpha\phi(u) + \delta\phi(v)\big)\,\partial_v\phi: = -\delta :\partial_v\phi\psi_1:. \end{aligned} \tag{2.14}$$

Following Coleman [2] we introduce

$$j_\mu{}^5 = \frac{-\beta}{2\pi}\frac{\partial\phi}{\partial x^\mu}, \quad j^\mu = \varepsilon^{\mu\nu} j_\nu{}^5 \qquad (\varepsilon_{01} = \varepsilon^{10} = 1) \tag{2.15}$$

which are obviously conserved and will be identified with the axial current and the current of the model, we can rewrite (2.14) as

$$i\gamma^\mu \frac{\partial}{\partial x^\mu}\psi = \frac{2\pi\delta}{\beta} :\gamma^\mu j_\mu\psi:. \tag{2.16}$$

is readily seen that the r.h.s. of (2.16) can also be written as

$$:j_\mu \psi: = \lim_{\varepsilon \to 0} \frac{1}{2} \{j_\mu(x+\varepsilon)\, \psi(x) + \psi(x)\, j_\mu(x-\varepsilon)\} \tag{2.17}$$

hich is the form given by KLAIBER [12] for the equation of motion.
he only remaining step in proving that the field (2.12) is indeed a solution of the Thirng model is to show that the currents (2.15) do generate γ^5 and gauge transformations.
o this end we recall that

$$\begin{aligned} \langle 0'|\, \phi(u)\, \phi(u')\, |0'\rangle &= -\frac{1}{4\pi} \log i(u-u') \\ \langle 0'|\, \phi(v)\, \phi(v')\, |0'\rangle &= -\frac{1}{4\pi} \log i(v-v') \end{aligned} \tag{2.18}$$

nd therefore

$$\begin{aligned} [\phi(x), \psi_1(x')] &= \left\{\frac{\alpha}{4}\, \varepsilon(u-u') + \frac{\delta}{4}\, \varepsilon(v-v')\right\} \psi_1(x') \\ [\phi(x), \psi_2(x')] &= -\left\{\frac{\delta}{4}\, \varepsilon(u-u') + \frac{\alpha}{4}\, \varepsilon(v-v')\right\} \psi_2(x'). \end{aligned} \tag{2.19}$$

rom (2.19) and (2.15) we get the following equal time commutation relations

$$\begin{aligned} [j^0(x), \psi(y)]_{\mathrm{E.T}} &= \frac{\beta}{4\pi}\, (\alpha-\delta)\, \psi(y)\, \delta(x_1-y_1) \\ [j^{50}(x), \psi(y)]_{\mathrm{E.T}} &= \frac{\beta}{4\pi}\, (\alpha+\delta)\, \gamma^5 \psi(y)\, \delta(x_1-y_1) \end{aligned} \tag{2.20}$$

hich show that j^0, j^{50} are indeed the local generator of gauge and γ^5 transformations.
t is convenient to normalize the currents in such a way that

$$\frac{\beta}{4\pi}\, (\alpha-\delta) = -1\,. \tag{2.21}$$

his normalization differs from the ones employed in [12] but is the conventional noralization for the generator of gauge transformation. The γ^5 transformations are then nconventionally normalized.
rom the well known properties of exponentials of free fields one computes

$$\langle 0|\, \prod_i e_{\alpha_i \delta_i}(x_i)\, |0\rangle = \delta_{\Sigma\alpha_i,0}\, \delta_{\Sigma\delta_i,0} \prod_i \prod_{j>i} [i(u_i-u_j)]^{\alpha_i\alpha_j/4\pi}\, [i(v_i-v_j)]^{\delta_i\delta_j/4\pi} \tag{2.22}$$

nd with $(\alpha_i \delta_i) = (\pm\alpha, \pm\delta)$, $(\mp\delta, \mp\alpha)$ and (2.12) obtain an arbitrary Wightman funcion for the Thirring model (The skeptical reader can also compare this result with LAIBER's [12] and convince himself that apart from a Klein transformation we do ndeed obtain the same Wightman functions).
rom the Wightman functions one can now read off the scale dimension and the Lorentz pin (there is no intrinsic spin in 2 dimensions!) of the field

$$d_\psi = \frac{\alpha^2+\delta^2}{8\pi}, \quad s = \frac{\alpha^2-\delta^2}{8\pi} \tag{2.23}$$

so that one recovers the whole two parametric manifold of Klaiber's solutions. In particular for $\delta = 0$, $\alpha^2 = 4\pi$ one has "bosonized" the free canonical spinor field.
Looking now at short distance expansions one has with (2.12)

$$\begin{aligned}
\psi_1^*(u, v)\,\psi_1(0) - \text{v.e.v.} \underset{u,v\to 0}{\sim} \frac{-i}{(iu)^{(\alpha^2/4\pi)}\,(iv)^{(\delta^2/4\pi)}\,2\pi}\,\{a u\,\partial_u\phi + \delta v\,\partial_v\phi\} \\
\psi_2^*(u, v)\,\psi_2(0) - \text{v.e.v.} \underset{u,v\to 0}{\sim} \frac{-i}{(iu)^{(\delta^2/4\pi)}\,(iv)^{(\alpha^2/4\pi}\,2\pi}\,\{\delta u\,\partial_u\phi + \alpha v\,\partial_v\phi\}
\end{aligned} \tag{2.24}$$

which means that as expected the current (pseudo current) are the leading q number terms in the Wilson expansion of $\bar\psi\gamma^\mu\psi$ and $\bar\psi\gamma^\mu\gamma^5\psi$, and

$$\psi_1^*(u, v)\,\psi_2(0) \underset{u,v\to 0}{\sim} \frac{[-x^2 + i\varepsilon x_0]^{(\alpha\delta/4\pi)}}{2\pi}\,{:}\exp i(\alpha + \delta)\,\phi{:} \tag{2.25}$$

which will play an important role in Coleman's equivalence [*2*].
In what follows we will be particularly intersted in the $s = 1/2$ solutions of the Thirring model; with (2.21) and (2.23) this gives

$$\delta - \alpha = \frac{4\pi}{\beta}, \qquad \alpha + \delta = -\beta. \tag{2.26}$$

It will also be useful to rewrite (2.12) in a way that does not rely on the u, v decomposition peculiar to a free massless field, so that for spin 1/2 one has [*15*]

$$\psi(x) = \frac{e^{(i\pi/4)\gamma^5}}{\sqrt{2\pi}}\,{:}\exp + i\beta\gamma^5\phi(x) + \frac{2\pi}{\beta}\,i\int_x^\infty \dot\phi(x')\,dx'{:}\begin{pmatrix}1\\1\end{pmatrix}. \tag{2.27}$$

In going from (2.12) to (2.27) we neglected $\phi(u = \infty) - \phi(v = -\infty)$ which is not zero but is proportional to the pseudo-charge (cf. (2.15) and (2.20)). Expressions (2.12) and (2.27) differ therefore by a Klein transformation which for $s = 1/2$ is just the right transformation to produce anti-commutation between the ψ_1 and ψ_2 components at space-like separations in (2.27) whereas in (2.12) they would commute.
The feature that has to be stressed in the "bosonization" of the Thirring model is not the amusing two dimensional pathology (cf. subsection 2c) of being able to write fermions in terms of bosons but the rather remarkable fact that buried within the theory of a neutral massless free field one has charged states, whose dynamics is described by the Thirring model. More precisely, besides the usual vacuum sector of the free theory one has charged sectors whose charge is obtained from an identically conserved current (2.15). Charge conservation is not a consequence of any Noether Symmetry of the Lagrangean but is due to the existence of finite energy solutions of the wave equation which do not vanish at spatial infinity and which manifest themselves in the quantized theory as inequivalent representations (sectors) of the free field algebra [*8*].
It is for the hidden richness which might exist in the inequivalent representations of a field theory that one should be looking for in a more realistic situation.

b) Coleman's Equivalence

The massless Thirring model discussed in subsection 2a is a scale invariant theory with anomalous dimensions. It is therefore natural to regard it as the high-energy asymptote of a massive model [*16*], which would be despite its dimension a much more interesting model.

Unfortunately, up to recently investigations of the massive Thirring model were limited to perturbation theory in the coupling constant. In the end of 1974 COLEMAN established [2] a remarkable correspondence between the massive Thirring and the Sine-Gordon theories. This means that in the same way as the massless Thirring model is related to the many inequivalent representation of the free scalar massless field the massive Thirring model will emerge from the many inequivalent sectors of the Sine-Gordon theory.
To understand how this comes about let us regard the massless Thirring as a short distance fixed point [17] of a broader class of theories which will have a energy momentum tensor given by

$$T^{\mu\nu} = T^{*\mu\nu} + \delta T^{\mu\nu} \tag{2.28}$$

where the $T^{*\mu\nu}$ is the fixed point energy momentum tensor which, due to the fact that the massless model has been entirely written is terms of a massless scalar field, is simply the energy-momentum tensor of a scalar massless field,

$$T^{*\mu\nu} = :\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\nu} - g^{\mu\nu}\left(\frac{\dot{\phi}^2 - (\nabla\phi)^2}{2}\right): \tag{2.29}$$

and $\delta T^{\mu\nu}$ a perturbation around the fixed point.
In order that the theory described by $T^{\mu\nu}$ has the massless Thirring model as a short distance fixed point the scale dimension of $\delta T^{\mu\nu}$ must be less than that of $T^{*\mu\nu}$ so that for short distances the perturbation becomes increasingly negligible:

$$\dim \delta T^{\mu\nu} < 2. \tag{2.30}$$

If inequality (2.30) is violated one will be for short distances driven away from the fixed point and the theory described by $T^{\mu\nu}$ (if it exists) can in no sense be considered as a perturbation of the massless Thirring model (In statistical mechanics where there is a natural lattice cut-off such a theory would correspond to a critical theory which is driven to the fixed point at large distances).
The simplest perturbation one can think off satisfying (2.70) is a mass perturbation $\bar{\psi}\psi$. From (2.25) we see that the renormalized operator corresponding to such a mass perturbation is given by

$$N(\bar{\psi}\psi) = \frac{1}{\pi} :\cos\beta\phi: \tag{2.31}$$

$$T^{\mu\nu} = T^{*\mu\nu} - \frac{\delta m}{\pi} :\cos\beta\phi: g^{\mu\nu}. \tag{2.32}$$

Since the scale dimension of $N(\bar{\psi}\psi)$ can be easily computed to give

$$d_{\bar{\psi}\psi} = \frac{\beta^2}{4\pi}, \tag{2.33}$$

consistency with (2.30) requires

$$\beta^2 < 8\pi. \tag{2.34}$$

Once one has written the energy-momentum tensor of the massive Thirring model in terms of the field ϕ one immediately recognizes that one is dealing with a sine-Gordon theory. In the sense of a mass perturbation around the massless Thirring model one arrives at a "bosonization" of the massive Thirring model with (2.31) and [2, 15]

$$\psi(x) = :\exp i\,\frac{\beta}{2}\,\gamma^5\phi(x) + \frac{2\pi}{\beta}\,i\int_{x_1}^{\infty}\dot{\phi}(x')\,dx_1'\,:\,\frac{e^{i(\pi/4)\gamma^5}}{\sqrt{2\pi}}\begin{pmatrix}1\\1\end{pmatrix} \tag{2.35}$$

$$j^\mu = -\frac{\beta}{2\pi}\,\varepsilon^{\mu\nu}\,\frac{\partial\phi}{\partial x^\nu} \tag{2.36}$$

where ϕ now is a solution of

$$\Box\phi + \frac{\beta\delta m}{\pi}\,:\sin\beta\phi: = 0\,. \tag{2.37}$$

The steps involved in a mass perturbation are quite formal though, and it is desirable to have a critical look at the final results; (a number of rigorous results have been obtained in the meantime [*18*] by J. FRÖHLICH).

To begin with let us remark that in two dimensions any conserved current can be written in the form (2.36) [*13, 19*].

This a trivial result for a classical current, but in two dimensions is also valid for a quantized theory with ϕ a local field. In higher dimensions one can always write a conserved classical current as $j^\mu = \partial_\mu F^{\mu\nu}$. Locality of the field $F^{\mu\nu}$ however can only be achieved if the theory contains "photons" [*20*]. The fact that there are no photons in two dimensions allows for the local integrability of ϕ

$$\phi(x) \stackrel{\text{def}}{=} \frac{2\pi}{\beta}\int_{-\infty}^{x} j_\mu(x')\,dx'^\mu\,, \qquad (dx'^\mu\,dx_\mu' < 0)\,. \tag{2.38}$$

Locality of the field ϕ defined by (2.38) is obvious from the path independence of the r.h.s. of (2.38). $\phi(x)$ measures the charge to the left of x. If the current belongs to a theory which is scale invariant at small distances the Schwinger term in the equal time commutator of j^0 with j^1 is finite and one can adjust β so that ϕ satisfies canonical equal time commutation relations

$$[\phi(x), \dot{\phi}(x')]_{\text{E.T}} = i\delta(x_1 - x_1')\,. \tag{2.39}$$

Consider now the equation of motion for ϕ

$$\Box\phi = F(\phi)\,. \tag{2.40}$$

Commuting (2.38) with a charged field and using (2.40) one finds the periodicity condition

$$F\left(\phi + \frac{2\pi}{\beta}\right) = F(\phi)\,. \tag{2.41}$$

The simplest solution to (2.41) is the sine-Gordon theory. For β sufficiently small one can introduce higher harmonics corresponding to perturbations other than the mass term that are asymptotically soft [*20*].

As in the case of the massless Thirring model the existence of charged sectors is related to the existence of solutions of the wave equation which do not vanish at spatial infinity with

$$Q = \frac{2\pi}{\beta}\left(\phi(\infty) - \phi(-\infty)\right). \tag{2.42}$$

The existence of such (finite energy) solutions of (2.40) and their interpretation as particles in a quantized theory was pointed out in [3, *22*]. Classically they are called solitons.

Let us look now more carefully into the equation of motion (2.37). We should first ask ourselves what do we mean by :: in a interacting theory. The simplest answer and the one adopted in [*2*, *15*] is to define it via Wick's theorem by subtracting the singularity of the free two point function, for instance

$$:\phi^2(x): = \lim_{\varepsilon\to 0}\left\{\phi(x)\,\phi(x-\varepsilon) + \frac{1}{4\pi}\log\varepsilon^2\right\}. \tag{2.43}$$

Is this a consistent prescription? Will it lead to a well defined equation of motion and finite expressions for (2.31) and (2.35)?

The minimal consistency requirement is that the theory described by (2.37) should be asymptotically free. If this is the case the leading singularities of $:\sin\beta\phi:$ can be computed from the massless free theory

$$\underset{\varepsilon\to 0}{\langle 0|}:\sin\beta\phi(\varepsilon):\ :\sin\beta\phi(0):|0\rangle \sim \frac{1}{(\varepsilon^2)^{(\beta^2/4\pi)}} \tag{2.44}$$

and therefore writing a Lehmann-Källen representation for the two point function

$$\langle 0|\,\phi(x)\,\phi(0)|\,0\rangle = \int^{\infty} \varrho(\mu^2)\,\Delta^+(x-y,\mu^2)\,d\mu^2 \tag{2.45}$$

we have from (2.44) and (2.37)

$$\varrho(\mu^2) \sim (\mu^2)^{(\beta^2/4\pi)-3}. \tag{2.46}$$

As expected this is only compatible with the normalization condition (2.39) which implies

$$\int^{\infty} \varrho(\mu^2)\,d\mu^2 = 1 \tag{2.47}$$

if the inequality (2.34) is satisfied.

Consider now

$$\langle 0|\,[-\Box\phi(x),\dot\phi(y)]_{\mathrm{E.T}}\,|0\rangle = i\,\langle 0|\,\frac{\delta m}{\pi}\,\beta^2\,:\cos\beta\phi(x):|0\rangle\,\delta(x_1-y_1)$$

$$\simeq i\int^{\infty} (\mu^2)^{(\beta^2/4\pi)-2}\,d\mu^2\delta(x_1-y_1) \tag{2.48}$$

which shows that if $\beta^2 \geq 4\pi$ Wick ordering will not be enough define the mass operator (2.31). Since $\beta^2 = 4\pi$ corresponds to the free massive model and the coupling constant defined from Eq. (2.16) is

$$G = -\frac{2\pi\delta}{\beta} = \pi\left(1 - \frac{4\pi}{\beta^2}\right) \tag{2.49}$$

we see that for repulsive coupling a more refined definition for the normal product is required. Lehmann and Stehr as well as Schroer and Truong [*21*] have investigated the free massive Thirring model using a normal product definition in which the true n point functions are subtracted. In this way they are led to a form of the equation of motion (2.37) which does not exibit explicitly the periodicity (2.41). An alternate procedure which keeps the periodicity condition will by highly desirable and can be probably achieved along the following lines: to begin with let us identify the source of our troubles for $4\pi \leqq \beta^2 < 8\pi$. Equation (2.31) for the correspondence between the mass operator and $\cos \beta\phi$ has been obtained on the basis of the short distance expansion (2.25) valid for the massless theory. The reason that the usual leading a number singularity is missing from (2.25) is the γ^5 invariance of the massless theory, allowing then for a multiplicative renormalization of the mass operator. In the massive theory however we will have a c-number singularity which although softer than $2d\psi$ might be strong enough to require a subtractive besides a multiplicative renormalization. For clarity let us start with the case of the free massive Thirring model. Direct computation gives

$$\bar{\psi}(x+\varepsilon)\,\psi(x) \underset{\varepsilon\to 0}{\sim} \frac{m}{\pi}\log \varepsilon_1{}^2 + N[\bar{\psi}\psi] \tag{2.50}$$

where in this case $N[\bar{\psi}\psi]$ is the finite operator defined by Wick ordering with respect to the fermions. From (2.50) we immediately see that the c number term although missing for $m = 0$ appears as a subtractive renormalization for $m \neq 0$. In the general case we will have

$$\bar{\psi}(x+\varepsilon_1)\,\psi(x) \sim f(\varepsilon_1) + \frac{1}{(\varepsilon_1{}^2)^{[(4\pi/\beta)^2-\beta^2]/16\pi}}\, N[\bar{\psi}\psi] \tag{2.51}$$

with $N[\bar{\psi}\psi]$ a finite operator normalized in such a way as to have zero vacuum expectation value and from (2.35), (2.48) $f(\varepsilon)$ should be given by

$$f(\varepsilon_1) = \frac{\pi}{(\varepsilon_1{}^2)^{[(4\pi/\beta)^2-\beta^2]/16\pi}\,.\,\delta m\beta^2}\int^{\varepsilon_1^{-2}} s\varrho(s)\,ds \tag{2.52}$$

with $\varrho(s)$ given by (2.45). The ansatz (2.51), (2.52) leads to:

a) for $\beta^2 < 4\pi$ the correspondence (2.31) is maintained up to a harmless additive constant.
b) for $\beta^2 = 4\pi$ one must subtract a logaritmically divergent counter term from $:\cos\sqrt{4\pi}\,\phi:$ in order to have a well defined operator in accordance with (2.50).
c) for $4\pi \leqq \beta^2 < 8\pi$ an infinite subtraction must be made. The singularity of $f(\varepsilon)$ is always less than $2d\psi$ corresponding to the vanishing of the c number term in the asymptotic γ^5 invariant theory.

All this leads us to propose as a correct definition of normal product valid in the whole range $0 \leqq \beta^2 < 8\pi$

$$N[e^{i\beta\,\phi(0)}] = \lim_{f\to\delta} :e^{i\beta\,\phi(f)}: - \langle 0|:e^{i\beta\,\phi(f)}:|0\rangle. \tag{2.53}$$

Definition (2.53) would leave unchanged the since-Gordon equation (2.27) as a result of the antisymmetry of the sine. It is a tempting conjecture, in so far unproven even far the free $\beta = \sqrt{4\pi}$ case, that the only operator of the sine-Gordon theory which is not rendered finite by Wick ordering is the mass operator.

For $\beta^2 \ll 1$ (weak coupling regime for the sine-Gordon theory strong attractive coupling for the Thirring model!) a number of very interesting results have been obtained using semi-classical methods [*3*, *22*], e.g. JACKIW [*40*].

c) Spin and Statistics in Two Dimensions [23]

It can be readily recognized that the essential feature of two dimensional world which allows one to "bosonize" fermions is the absence of intrinsic spin, and therefore no true Spin-Statistics Theorem. What one normally calls "spin" in two dimensions is the Lorentz spin i.e., the transformation properties of the wave-function or field operator under Lorentz transformations. As long as there exist one particle states one can (trivially) carry through Wigner's [24] famous analysis for a two dimensional space-time to conclude that one particle states can always be chosen to transform as scalars

$$U(\lambda)\,|p_0p_1\rangle = |\cosh\lambda p_0 + \sinh\lambda p_1,\, \cosh\lambda p_1 + \sinh\lambda p_0\rangle. \tag{2.54}$$

Clearly one is also free to introduce an equivalent description with

$$|p_0p_1\rangle_s = \left(\frac{p_0 - p_1}{m}\right)^s |p_0p_1\rangle \tag{2.55}$$

implying a "spin s" transformation law

$$U(\lambda)\,|p\rangle_s = e^{-s\lambda}|p_\lambda\rangle_s. \tag{2.56}$$

The possibility of assigning different Lorentz spins to the same state corresponds in 4 dimensions to the well known fact that a (free) particle of intrinsic spin s can be described equivalently by many relativistic wave equations transforming differently under the Lorentz group.

If there are zero mass states in the theory one can have a larger symmetry group such as the conformal group and this reflects itself in the fact that the different "spin" solution of the Thirring model correspond to different representations of the conformal group [9, 25].

For a massive theory however, the "spin" one assigns to the states is entirely a matter of convention. This is well known.

What is perhaps more surprising is that in a sense to be made precise below the statistics in a twodimensional field theory is also conventional. This is typically two dimensional and has no analogy in higher dimensions. In a field theory what one ultimately calls statistics is the statistics obeyed by the asymptotic (in the old L.S.Z. sense of the word) free particle states. Suppose for definitness those states were bosons

$$|p\rangle = a^+(p)\,|0\rangle, \qquad [a(p), a^+(p')] = \delta(p_1 - p_1')\,p^0 \tag{2.57}$$
$$[a(p), a(p')] = [a^+(p), a^+(p')] = 0.$$

Consider now

$$b^+(p) = a^+(p)\exp\left(-\,i\pi\int_{p_1}^{\infty} a^+(p')\,a(p')\,\frac{dp_1'}{p_0}\right). \tag{2.58}$$

The b's satisfy canonical anticommutation relations i.e. they are fermion operators. This simply means that there is a one to one mapping between antisymmetrical and symmetrical p-space wave functions

$$f_A(p, p') = \varepsilon(p_1 - p_1')\,f_s(p, p') \tag{2.59}$$

which allows one to interpret any bosonic state in terms of fermions and vice-versa. Although in higher dimensions similar mappings can be introduced they do not share with (2.58, 2.59) the property of being Lorentz invariant.

One could be formally tempted introduce in two dimensions a generalized continuou statistics (not parastatistics!) by replacing π in the exponent of the r.h.s. of (2.58) b an arbitrary number s $0 \leq s < 2\pi$. However if one demands in accordance with genera principles that the Fourier transform of the momentum space wave function describe probabilities of (approximate) position measurements the simultaneous requirement

$$|\tilde{f}(x, x')|^2 = |\tilde{f}(x', x)|^2; \qquad |f(p, p')|^2 = |f(p', p)|^2 \tag{2.60}$$

restricts our choice of s to be 0 or π.

Our assertion that the assignment of Bose or Fermi Statistics to the particle states of given two dimensional quantum field theory is entirely conventional seems to contradic the well known fact (valid also in a two dimensional world) that a periodic table o elements requires fermions.

The apparent paradox is resolved by realizing that to find the energy levels of an aton with a given local potential one needs an at least approximate notion of localization (Allowing for the highly non-local interactions induced by the mapping (2.59) one ca have a Bose system exhibiting the same energy levels as a Fermi system with loca interactions.)

In a field theoretical context the notion of localization comes from the fact that a fiel $\psi(x)$ is supposed to create "something" in the vicinity of x. The problem in two dimen sions is that there are many fields, local with respect to themselves but not with respec to each other that carry different notions of localization, and different statistics.

Let us illustrate this point in the massive Thirring model. The conventional descriptio is in terms of a fermion field ψ which will via an L.S.Z. asymptotic condition lead to particle interpretation in terms of fermions. On the other hand using ϕ as given b equation (2.38) on can introduce a Bose field

$$\psi'(x) = N\{e^{-i(\beta/2)\,\Phi(x)\gamma^5}\psi(x)\} \tag{2.61}$$

which will lead to a description of the Thirring model in terms of bosons. In order t avoid the technical problem of defining the correct normal product in (2.61) one coul use an operator afiliated with a finite region

$$\psi_f' = e^{-i(\beta/2)\,\Phi(f)}\psi(f) \tag{2.62}$$

$$\int f(x)\, d^2x = 1$$

which will commute with its translate for sufficiently large space like separations an used as the interpolating operator in a Haag-Ruelle collision theory will lead to Bose statistics for the asymptotic free particles of the model.

One could try to argue that in the Thirring model the Fermi description is prefere because in this case we have formally an underlying canonical structure for the fermio fields.

Besides being mathematically unprecise since a true canonical structure exists onl for free fields the physical meaning of a such a guiding principle is quite obscure.

As a result we see that in a two dimensional field theory one has the freedom of assignin any statistics to the asymptotic free particles depending on what field we choose t represent the localization properties of the theory.

3. The Schwinger Model

The Schwinger model [5, 6], quantum-electrodynamics of massless fermions in tw dimensional space-time, is a standard soluble example of a theory where there are n particles carrying the quantum-numbers one would associate with a continuous sym-

metry group of the Lagrangean, having been thus proposed [26] as a prototype model for Quark Confinement.
It can be formulated in terms of the equations of motion

$$i\gamma^\mu \frac{\partial}{\partial x^\mu} \psi(x) + \frac{e}{2} \lim_{\varepsilon\to 0} \{A_\mu(x+\varepsilon)\, \gamma^\mu\psi(x) + \gamma^\mu\psi(x)\, A_\mu(x-\varepsilon)\} \tag{3.1}$$

$$\frac{\partial}{\partial x^\nu} F^{\mu\nu}(x) = -ej^\mu(x) \tag{3.2}$$

where

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$$

and $j^\mu(x)$ is the gauge invariant current

$$j^\mu(x) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon^2<0}} \left[\overline{\psi}(x+\varepsilon)\, \gamma^\mu\psi(x) - \langle 0|\, \overline{\psi}(x+\varepsilon)\, \gamma^\mu\psi(x)\, |0\rangle \left(1 - ie\varepsilon^\mu A_\mu(x)\right)\right]. \tag{3.3}$$

As in the case of the Thirring model we can look for solutions of (3.1) and (3.2) in the form exponentials of free fields

$$\psi(x) = \frac{(\mu e^\gamma)^{1/2}}{(4\pi)^{1/2}}\, e^{i(\pi/4)\gamma^5} :\exp i(\pi)^{1/2}\gamma^5\Sigma(x): \begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2} \end{pmatrix} \tag{3.4}$$

with θ_1, θ_2 arbitrary c number phases and μ the mass of the free field Σ.
Applying the Dirac equation to (3.4) and using

$$\gamma^\mu\gamma^5 = \varepsilon^{\mu\nu}\gamma_\nu$$

we are led to the identification

$$A^\mu(x) = -\frac{(\pi)^{1/2}}{e} \{\varepsilon^{\mu\nu}\, \partial_\nu\Sigma(x)\} \tag{3.5}$$

with (3.4) and (3.3) we readily find

$$j^\mu(x) = -\frac{1}{\sqrt{\pi}}\, \varepsilon^{\mu\nu}\partial_\nu\Sigma(x) \tag{3.6}$$

and therefore from (3.2) we obtain

$$\left(\Box + \frac{e^2}{\pi}\right)\Sigma(x) = 0 \tag{3.7}$$

$$\mu = e/\sqrt{\pi}.$$

This means that Σ is a free field of mass $e/\sqrt{\pi}$. Being a massive field the selection rules we had to introduce to properly define exponentials of massless fields in section 2 do not come into play now. This is related to the spontaneous breakdown of gauge and γ^5 invariance [6].
At first sight it seems surprising that our ansatz (3.4) which ought to represent a fermion field commutes with itself at space-like separations. We should remember however that $\psi(x)$ is not a gauge invariant operator and therefore its commutation relations depend on the q-number gauge employed.
It is possible to obtain (3.4) by means of a gauge transformation [6] starting from the more conventionally looking Schwinger solutions [5].
The "observable" content of two dimensional quantum electrodynamics should be entirely given by the algebra of gauge invariant operators. Besides the electric field

and the current one can introduce the properly defined gauge invariant bilocals [6] corresponding to the formal bilocals $\psi(x)\,\{\exp ie \int_x^y A^\mu(z)\,dz_\mu\}\,\psi^*(y)$, which in our gauge can be written as

$$T(x,y) = N(x-y)\;{:}\exp i\sqrt{\pi}\,\{\gamma_x{}^5\Sigma(x) - \int_x^y \varepsilon^{\mu\nu}\,\partial_\nu\Sigma(z)\,dz_\mu - \gamma_y{}^5\Sigma(y)\}{:} \tag{3.8}$$

with the normalization matrix $N(z)$ given by

$$N(z) = -\frac{1}{2\pi}\begin{pmatrix} \dfrac{-i}{(z_0+z')} & \dfrac{\mu e^{i(\theta_1-\theta_2)+\gamma}}{2} \\ \dfrac{\mu e^{-i(\theta_1-\theta_2)+\gamma}}{2} & \dfrac{-i}{(z_0-z')} \end{pmatrix}. \tag{3.9}$$

With this normalization one ensures on one hand that the bilocals transform as if they were bilinear in "spin 1/2" fields and on the other that the gauge invariant current is simply given by

$$j^\mu(x) = -\lim_{\varepsilon\to 0}\Big\{\mathrm{Tr}\,\big(\gamma^0\gamma^\mu T(x+\varepsilon,x)\big) - \langle 0|\,\mathrm{Tr}\,(\gamma^0\gamma^\mu T(x+\varepsilon,x)\,|0\rangle\Big\}. \tag{3.10}$$

One can formally rewrite $T(x,y)$ for equal times as

$$T(x,y) = N(x-y)\;{:}\psi_c(x)\,\psi_c{}^*(y){:} \tag{3.11}$$

with

$$\psi_c(x) = e^{i\pi/4\gamma^5}\,\frac{(\mu e^\gamma)^{1/2}}{(4\pi)^{1/2}}\;{:}\exp i\sqrt{\pi}\left\{\gamma^5\Sigma(x) + \int_x^\infty \dot{\Sigma}(x')\,dx'\right\}{:}\begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2}\end{pmatrix} \tag{3.12}$$

with $\psi_c(x)$ differing from $\psi(x)$ by a q-number gauge transformation. In this new gauge we have

$$A_1 = 0$$

$$A_0 = \int_x^\infty E(x')\,dx' \tag{3.13}$$

i.e. the Coulomb gauge.

As in (2.27), (3.12) represents an anticommuting "field". Contrary to (2.27) the expression for the Coulomb gauge fermion operator cannot be given a precise meaning. The reason for that is that the equation of motion for Σ (3.7) violates the periodicity condition (2.41), and therefore no charged fields can be introduced in the theory.

Physically [6, 26] one can understand this feature in the following way: $T(x,y)$ creates a charge dipole with an electric field in between (in accordance with Gauss law). $\psi_c(x)$ would correspond to a situation where one of the charges is removed to infinity. In two dimensional space time the growth of the Coulomb potential with increasing separations implies an infinite cost in energy to separate the pairs. As a result the physical state space does not contain any excitations corresponding to the original fermions.

The only physical excitation of the theory are the Σ mesons which can be viewed as a fermion-antifermion bound state.

Quantum electrodynamics in two dimensions contains solely the vacuum sector. The symmetries one reads off from the Lagrangean namely gauge and γ^5 invariance do not correspond to any quantum numbers in the physical state space. Gauge transformations of the first kind are generated by the current (3.6) which leads to an identically zero charge. On the other hand there is no conserved gauge invariant current generating γ^5 transformations as a result of the two dimensional analog of the axial-vector anomaly [*27*] that is, with $j^{5\mu} = \varepsilon^{\mu\nu} j_\nu$ one finds

$$\partial_\mu j^{5\mu}(x) = \frac{e}{2\pi}\, \varepsilon_{\mu\nu} F^{\mu\nu}(x) \tag{3.14}$$

and the corresponding conserved non-gauge invariant pseudocurrent vanishes identically

$$\tilde{j^{5\mu}} = j^{5\mu} - \frac{e}{\pi}\, \varepsilon^{\mu\nu} A_\nu = 0. \tag{3.15}$$

One has therefore a spontaneous breakdown of both symmetries without any Goldstone bosons [*28*], which here however is intrinsic dispensing the use of any Higgs field [*29*]. The phases introduced in the r.h.s. of (3.4) characterize the different vacua corresponding to this spontaneous breakdown. Although the disappearance of quantum numbers is a common feature of any theory with a spontaneous symmetry breakdown, the peculiarity of two dimensional quantum-electrodynamics is the simultaneous disappearance of the particles that would carry those quantum numbers. This happens because gauge invariance of the second kind strongly restricts the class of allowed "observables" and physical states of the theory: the original fermions of Schwinger's solution [*5*] are simply gauged away.

To be more specific, in a non-gauge theory with a continuous spontaneous symmetry breakdown the charge operator Q cannot be defined [*30*]. One can however still recognize in the incoming and outgoing states the (no longer degenerate) multiplet structure of the theory. This is no longer the case in a gauge theory with spontaneous symmetry breakdown, as the Schwinger model clearly illustrates.

Although absent from the physical space, the original fermions make their reappearance of one considers the short distance behaviour of the Schwinger model. The leading singularities of Green's function of gauge invariant operators such as the current and the scalar and pseudoscalar densities

$$-\lim_{\varepsilon\to 0} \mathrm{Tr}\,\{\gamma^0 T(x+\varepsilon, x)\} = \frac{\mu e^\gamma}{2\pi} :\cos\left(\sqrt{4\pi}\,\Sigma(x) + (\theta_2 - \theta_1)\right): \tag{3.16}$$

$$-i\lim_{\varepsilon\to 0} \mathrm{Tr}\,\{\gamma^0\gamma^5 T(x+\varepsilon, x)\} = \frac{-\mu e^\gamma}{2\pi} :\sin\left(\sqrt{4\pi}\,\Sigma(x) + (\theta_2 - \theta_1)\right): \tag{3.17}$$

can be obtained by letting the mass of the Σ field tend to zero. This bring us back to the situation discussed in section 2 and therefore the short distance asymptote of two dimensional q.e.d. is nothing but the Thirring model with $\beta = \sqrt{4\pi}$, that is a free theory of charged massless fermions.

The fact that under short distance probing the theory behaves as if it contained particles which do not manifest themselves as physical states has been recognized by CASHER, KOGUT and SUSSKIND [*26*] as being precisely what one desires of a theory of quark confinement.

After this brief recapitulation of the massless Schwinger model we can follow COLEMAN, JACKIW and SUSSKIND [*11*], and in the same spirit as was done for the massive Thirring model introduce a fermion mass. In bosonic language, using the fact that the massless Schwinger model is isomorphic to the theory of a free Σ field with mass $\mu = e/\sqrt{\pi}$ and expression (3.16) for the mass operator we are imediately led to a theory described by the energy momentum tensor

$$T^{\mu\nu} = \partial^\mu \Sigma \partial^\nu \Sigma - \frac{g^{\mu\nu}}{2}\left(\partial^\Lambda \Sigma \partial_\Lambda \Sigma - \mu^2 \Sigma + \frac{\delta m \mu e^\gamma}{\pi} \cos\left(\sqrt{4\pi}\,\Sigma + (\theta_2 - \theta_1)\right)\right) \quad (3.18)$$

and therefore to the massive sine-Gordon theory:

$$(\Box + \mu^2)\,\Sigma + \frac{\delta m \mu e^\gamma}{\sqrt{\pi}} :\sin\left(\sqrt{4\pi}\,\Sigma + (\theta_2 - \theta_1)\right):. \quad (3.19)$$

As in the massless Schwinger model the absence of charge sectors can be seen from the fact that the explicit mass term in (3.19) violates the periodicity condition (2.41). On a classical level this results from the non-existence in general of classical finite energy solutions of (3.19) which have a different asymptotic behaviour for $x_1 \to \pm\infty$. For $\theta_2 - \theta_1 = \pi$ and δm sufficiently large there are however classical kinks [*3*] which do not in the quantized version lead to an additive charge quantum number (cf. section 4). As before we can introduce gauge invariant bilocals as in (3.8)

$$T(x, y) = N(x, y) :\exp i \sqrt{\pi} \left\{\gamma_x{}^5 \Sigma(x) - \int_x^y \varepsilon^{\mu\nu}\, \partial_\nu \Sigma(z)\, dz_\mu - \gamma_y{}^5 \Sigma(y)\right\}: \quad (3.20)$$

and with (3.10) we get the gauge invariant current

$$j^\mu = -\frac{1}{\sqrt{\pi}}\, \varepsilon^{\mu\nu}\, \partial_\nu \Sigma \quad (3.21)$$

in complete analogy to (3.6). We should however at once refrain from trying to carry over to the massive Schwinger model the remaining correspondences (3.4) and (3.5). They are clearly incompatible with Maxwell's equation if Σ satisfies (3.19). The reason is that the particular gauge were the solutions of the massless Schwinger model were obtained does not survive a mass perturbation. If one uses the freedom given by gauge transformations of the 2nd kind one can formally go over to the Coulomb gauge were (3.11), (3.12) and (3.13) hold and obtain then a formal bosonization of the coupled massive Dirac and Maxwell equations. This has at most a heuristic value since as in the massless case the Coulomb gauge operators do not exist.

A better insight into the behaviour of gauge variant operators which at the same time throws an additional light into both the problem of confinement and the structure of the Dirac equation of motion (which we did not discuss in connection with massive Thirring model), can be obtained by regarding quantum-electrodynamics both massless and massive as a limit of a vector-meson theory [*6, 31, 32*] (Thirring-Wess model). This means we explicitly break gauge invariance of the 2nd kind by a bare mass term μ_0 for the vector meson and then study the limit of $\mu_0 \to 0$. Maxwell's equation are then replaced by

$$\partial_\nu F^{\mu\nu} + \mu_0{}^2 B^\mu = -e j^\mu \quad (3.22)$$

with j^μ defined by the gauge invariant limit (3.10).

For the fermion field, which as long as $\mu_0 \neq 0$ will be a well defined operator we make the ansatz, suggested by the analogous expression in the Thirring-Wess [*26, 32*] model

$$\psi(x) = e^{i\pi/4\gamma^5} \left(\frac{\mu e^\gamma}{4\pi}\right)^{1/2} :\exp i \left\{\gamma^5 \frac{\beta}{2}\, \phi(x) + \frac{2\pi}{\beta} \int_x^\infty \dot{\phi}(x')\, dx' + \gamma^5 a \Sigma(x)\right\}: \begin{pmatrix}1\\1\end{pmatrix} \quad (3.23)$$

and for the vector meson field B^μ again by analogy we take

$$B^\mu = -\frac{1}{e}\left(a\varepsilon^{\mu\nu}\,\partial_\nu \Sigma + \frac{\beta}{2}\left(1 - \frac{4\pi}{\beta^2}\right)\varepsilon^{\mu\nu}\,\partial_\nu \phi\right). \tag{3.24}$$

a and β are constants to be determined in terms of μ_0 and e. In the Thirring-Wess model a simultaneous solution of the coupled Dirac and Proca equations is obtained if [6],

$$\left(\Box + \left(\mu_0{}^2 + \frac{e^2}{\pi}\right)\right)\Sigma = 0, \qquad \Box\phi = 0. \tag{3.25}$$

From (3.23), (3.20) we find that the mass operator is

$$-\lim_{\varepsilon\to 0} \mathrm{Tr}\,\{\gamma^0 T(x+\varepsilon, x)\} = \frac{\mu e^\gamma}{2\pi}\,:\cos(\beta\phi + 2a\Sigma): \tag{3.26}$$

(Implicit in our ansatz and for the remainder of this section we take $\theta_2 - \theta_1 = 0$).
The massive (in the sence of a fermion mass) Thirring-Wess model should therefore correspond to fields Σ and ϕ satisfying

$$(\Box + \mu^2)\,\Sigma + \frac{a\delta m\mu e^\gamma}{\pi}\,:\sin(\beta\phi + 2a\Sigma): = 0 \tag{3.27a}$$

$$\mu^2 = \mu_0{}^2 + \frac{e^2}{\pi}$$

$$\Box\phi + \frac{\beta\delta m\mu e^\gamma}{2\pi}\,:\sin(\beta\phi + 2a\Sigma): = 0. \tag{3.27b}$$

The gauge invariant current is given with (3.23) and (3.20)

$$j^\mu(x) = -\lim_{\varepsilon\to 0}\mathrm{Tr}\,\{\gamma^0\gamma^\mu T(x+\varepsilon, x)\} = -\frac{\beta}{2\pi}\,\varepsilon^{\mu\nu}\partial_\nu\phi - \frac{a}{\pi}\,\varepsilon^{\mu\nu}\,\partial_\nu\Sigma. \tag{3.28}$$

We shall now determine a and β using the Proca equation (3.22). With (3.24) and (3.28) one can rewrite (3.22) as

$$\Box\left(a\Sigma + \frac{\beta}{2}\left(1 - \frac{4\pi}{\beta^2}\right)\phi\right) + \mu^2 a\Sigma + \left\{\frac{\mu_0{}^2\beta}{2}\left(1 - \frac{4\pi}{\beta^2}\right) + \frac{e^2\beta}{2\pi}\right\}\phi = 0. \tag{3.29}$$

Comparing (3.27) with (3.29) we find out

$$\left(1 - \frac{4\pi}{\beta^2}\right) = -\frac{e^2}{\mu_0{}^2\pi}; \qquad a^2 + \frac{\beta^2}{4}\left(1 - \frac{4\pi}{\beta^2}\right) = 0; \qquad a = \frac{e}{\mu} \tag{3.30}$$

as a condition for satisfying Proca's equation. Note that (3.30) is independent of the fermion mass and coincides for zero fermion mass with the values given in [6]. It remains now to show that Dirac's equation for ψ given by (3.23) is compatible with the identification of B^μ given by (3.24) as a vector meson field coupled to the fermion field.
Writing [32]

$$\psi(x) = \lim_{\varepsilon\to 0}\,:e^{iA_\varepsilon(x)}: = \lim_{\varepsilon\to 0} Z'(\varepsilon)\,e^{iA_\varepsilon(x)} \tag{3.31}$$

where a smearing of radius ε around x has been introduced, and using

$$\delta e^{iA_\varepsilon(x)} = \int_0^1 e^{i\lambda A_\varepsilon(x)} \delta A_\varepsilon(x)\, e^{-i\lambda A_\varepsilon(x)}\, d\lambda e^{iA_\varepsilon(x)} \tag{3.32}$$

one readily gets applying Dirac's operator to (3.31)

$$i\gamma^\mu\, \partial_\mu \psi(x) + e :\gamma^\mu B_\mu(x)\, \psi(x):$$

$$- i\gamma^0 \int h_\varepsilon(y_1)\, dy_1 \int_0^1 e^{i\lambda A_\varepsilon(x)} \delta m \mu e^\gamma :\sin(\beta\phi(y) + 2a\Sigma(y): e^{-i\lambda A_\varepsilon(x)}\, d\lambda \psi(x) = 0 \tag{3.33}$$

with B^μ given by (3.24) and $h_\varepsilon(y_1) \xrightarrow[\varepsilon\to 0]{} \theta(y_1 - x_1)$. The last term on the l.h.s. of (3.33) can be rewritten [*15, 28*], using the fact that A_ε acts as a shift operator on ϕ, as an equal time commutator. Last term of (3.33) is

$$-\gamma^0 \frac{\delta m \mu e^\gamma}{2\pi} \left[\int dy_1 :\cos\left(\beta\phi(y) + 2a\Sigma(y)\right):, \psi(x)\right]_{\text{E.T}} \tag{3.34}$$

in perfect agreement with what one expects from the Hamiltonian form of the equation of motion. Using now the short distance singularities in the product of the mass and ψ operators one finally arrives at

$$i\gamma^\mu\, \partial_\mu \psi(x) + e :B_\mu(x)\, \gamma^\mu \psi(x): + \delta m \psi(x) = 0 \tag{3.35}$$

which completes our discussion of the massive Thirring-Wess model.

Notice at this point that the reason we get an explicit mass term in (3.35) is due to the fact that the scale dimension of our mass operator with β and a given by (3.30) is one. A similar calculation for $\beta < \sqrt{4\pi}$ in the Thirring model would lead to a vanishing equal time commutator in (3.34) and no explicit mass term in the Dirac equation of motion. This does not mean that the massive Thirring model for $\beta < \sqrt{4\pi}$ is really massless but simply reflects the fact that in quantum field theory one should not expect that the form of the renormalized equations of motion uniquely defines the theory. It is clear that in general $:\gamma_\mu j^\mu \psi:$ does contain hidden mass term that is $:\gamma_\mu j^\mu \psi: = N(\gamma_\mu j^\mu \psi) + c\psi$ with $N(\gamma_\mu j^\mu \psi)$ so normalized that $\langle 0|\, N(\gamma_\mu j^\mu \psi)\, |1\rangle = 0$.

After this small detour into the Thirring model let us come back to the problem a hand that is recovering the massive Schwinger model as an limit when $\mu_0 \to 0$ of the massive Thirring-Wess model.

First notice that as long as $\mu_0 \neq 0$ one has a local solution of ones field equations in a positive definite Hilbert space. The fermions are not confined since the periodicity of the equation (3.27b) for the ϕ field allows for infinite line integrals in our ansatz (3.23) and therefore charge sectors. When $\mu_0 \to 0$, $\beta \to 0$ and (3.23) becomes ill defined. It is suggestive that the divergent part has the form of a gauge term and therefore, should not participate in gauge invariant quantities. Indeed considering

$$T(x, y) = N(x - y) :\exp i\{A(x) + e \int_x^y B^\mu(z)\, dz_\mu - A(y)\} \tag{3.36}$$

we recover in the limit $\mu_0 \to 0$ our old finite bilocal (3.20). The ϕ field which in this limit becomes a massless free field completely decouples from gauge invariant operators, such as the bilocals the current and the $F^{\mu\nu}$. This decoupling explains how charge sectors

disappear in the limit. The charged states of the Thirring-Wess model become orthogonal to all the physical states. The ϕ field is pure gauge and any fermion build with it is necessarily aphysical. The physical fermions became confined.
Although it is clear that the physical origin of confinement is the growing Coulomb potential it is also important to stress that in a field theoretical context particles have a clear meaning as such only as asymptotic incoming and outgoing states. Field theoretical confinement is much more dramatic than its classical counterpart: in the later by pumping an (incredibly large) amount of energy into a bound pair one could have the components separated by a macroscopic distance and identified as charged fermions; what the Schwinger model teaches us is that in a field theoretical description of confinement the confined objects are simply not there. One can however always investigate the charge distribution of a dipole

$$\langle d|\, j^0(x)\, |d\rangle = \frac{1}{\sqrt{\pi}} \langle d|\, \partial_1 \Sigma(x)\, |d\rangle \tag{3.36}$$

with

$$|d\rangle \sim \exp -i\sqrt{\pi} \left[\int\limits_{x_1}^{y_1} \dot{\Sigma}(y_1, 0)\, dy_1 \right] |0\rangle \tag{3.37}$$

and a smearing around x_1 and y_1 being understood. In the massless Schwinger model Σ is a free field and one immediately sees that the total charge of the electric poles at the end of string oscilates periodically in time with a frequency $e/\sqrt{\pi}$, with an oscilating current flowing along the string. This can be understood as coming from vacuum polarization effects [*26*] with the electric energy of the dipole $e^2|x_1 - y_1|$ being used up to create virtual pairs. The massless nature of the virtual fermions accounts for the fact that the charge distribution is unstable no matter how small $|x_1 - y_1|$. In the massive Schwinger model on the other hand one expects that for a finite fermion mass δm there should be a critical length $e^2|x_1 - y_1| \sim 2\delta m$ below which there is charge stability. In such a case one can understand that a quantum mechanical description in terms of "particles" interacting via a growing potential can work for low lying bound states as a first approximation to the field theoretical problem.

4. Remarks on Charge Sectors

A mathematically precise framework for the construction of the charge sectors of an observable algebra has been set up by Doplicher, Haag and Roberts [*33*] and was applied in [*8, 18, 34*] to two dimensional models. In this section we will present an heuristic approach to the construction of charge raising operators for the various models of sections 2 and 3. In the course of this section the connection between existence or non-existence of charged states and the periodic or non-periodic nature of the field equations for the underlying scalar fields, will be clarified.
As in equation (3.37) consider a (smeared) dipole state

$$|d\rangle = \exp \{i\alpha \int h(x^1, y^1 \mid \phi z^1)\, \dot{\varphi}(z^1, 0)\, dz^1\} \,|0\rangle \tag{4.1}$$

where $h(x^1 y^1 \mid z^1)$ is a smoothed out $\theta(z^1 - x^1)\,\theta(y^1 - z^1)$ and φ will stand generally for either ϕ or Σ of the preceding sections. In order to have the exponential as a bona fide unitary operator an additional time smearing is required whenever the scale dimension of the source of the φ field is larger or equal to one. In the Thirring model this means $\beta \geq \sqrt{4\pi}$ and follows from the fact that in this case (2.46) requires that $\dot{\phi}$ be smeared in space and time to be a well defined operator.

The dipole state (4.1) represents a pair of negative and positive charges, localized around x^1 and y^1 at time zero, with respect to the charge density operator

$$j^0(x) = \frac{1}{\alpha}\,\partial_1\varphi(x)\,. \tag{4.2}$$

One formally obtains a charged state by letting $y_1 \to \infty$ or $x_1 \to -\infty$. In this limit the unitary operator ceases to operate in the original Hilbert space (vacuum sector) and plays the role of an intertwining operator between inequivalent representation of the observable algebra (charge sectors).
The Hamiltonian of the field being given by

$$H = \frac{1}{2}\int dx^1 : \dot{\varphi}^2 + (\nabla\varphi)^2 + F(\varphi): \tag{4.3}$$

one can compute the energy difference between the dipole state and the vacuum

$$\langle d|\,H\,|d\rangle - \langle 0|\,H\,|0\rangle = |x^1 - y^1|\,\langle 0|\,:F(\varphi - \alpha): - :F(\varphi):|0\rangle + E(x) + E(y) \tag{4.4}$$

where $E(x)$ and $E(y)$ are contributions coming from the neighborhood of x^1 and y^1 representing localization energies, and use has been made of the fact that $\dot{\varphi}$ acts as a shift operator on φ.
Equation (4.4) shows the fundamental difference between the various models discussed in the preceding sections. For the massless Thirring model $F \equiv 0$ and one will obtain a finite energy state by letting the two charges in the dipole to become infinitely separated for any value of α. One has therefore a continuous infinity of inequivalent representations corresponding to the continuous spin solutions of KLAIBER [*12*, *8*].
If F is periodic with period α one has again a finite energy state for an infinite separation of the pair in the dipole state. Successive application of the exponential in the r.h.s. of eq. (4.1) to the vacuum will give rise in the limit $|x_1 - y_1| \to \infty$ to inequivalent representations labelled by an integer number. This corresponds to the situation found in the massive Thirring and Thirring-Wess models with φ standing for ϕ. In the Schwinger model on the other hand, the non-periodic nature F implies in (4.4) that no finite energy charged state exists. The formal limit $|x^1 - y^1| \to \infty$ in this case leads us to the formal Coulomb gauge formulation of the Schwinger model, whose pathological features [*6*] arise from the fact that it is based on infinite energy states.
If in (4.1) one uses instead of $h(xy\,|\,z)$ an arbitrary function vanishing for $z \to -\infty$ and equal to 1 for $z \to +\infty$ an variational ansatz in (4.4) one makes easy contact with the semi-classical approaches of [*3*, *22*].
One should be warned at this point of the fact that, although for semi-classical computations one can always use coherent states of the form (4.1), the existence of charged states as limits of dipole states of the form (4.1) requires a periodic F. This is of interest in connection with the "kink" of GOLDSTONE and JACKIW [*3*] where non trivial sectors exist for a non-periodic F satisfying a symmetry condition

$$F(\varphi) = F(-\varphi)$$

which is spontaneously broken

$$\langle\varphi\rangle = \varphi_0 \neq 0\,.$$

Although one can always adjust α in such a way that in (4.4)

$$\langle 0|\,:F(\varphi - \alpha): - :F(\varphi):|0\rangle = 0 \tag{4.5}$$

so that the expectation value of the energy remains finite as $|x^1 - y^1| \to \infty$, it is readily seen that in the "kink" case since

$$:F(\varphi - \alpha): \neq :F(\varphi):$$

for any α the energy fluctuations in the state $|d\rangle$ will diverge in the limit

$$\langle d| (H - \langle 0| H |0\rangle)^2 |d\rangle \to \infty .$$

$$|x^1 - y^1| \to \infty .$$

This means that contrary to the soliton case the creation operator for a kink cannot be written as

$$\psi(x) \sim \exp i\alpha \int_{x_1}^{\infty} \dot{\varphi}(z^1, 0)\, dz^1 . \tag{4.6}$$

In order to construct a kink operator one should be guided by the symmetry of the potential $F(\varphi)$. Such an operator must act as the identity on fields located at left spatial infinity and as a transformation $\phi(x) \to -\phi(x)$ for $x^1 \to \infty$.
Introducing for time $t = 0$ the canonical decomposition of φ, $\dot{\varphi}$ in terms of creation and anihilation operators one goes over to a complex field χ

$$\chi(x^1, 0) = \frac{1}{(2\pi)^{1/2}} \int dk^1 a(k^1)\, e^{ik^1x^1}.$$

The unitary operator

$$U(x^1, y^1) = \exp i\pi \int h(x^1, y^1 \mid z^1)\, \chi^*(z^1, 0)\, \chi(z^1, 0)\, dz^1$$

generates the transformation $\chi \to -\chi$ in the interval (x^1, y^1) and applied to the vacuum produces, the dipole analogue for the kink problem. Although $\varphi, \dot{\varphi}$ are not strictly local with respect to χ they are quasi-local so that in the limit $y^1 \to \infty$, $U(x^1, \infty)$ effectively acts on φ's very much to the right as a phase-space rotation of angle π, $\varphi \to -\varphi$. A natural candidate for the kink operator is therefore [35]

$$\psi_{\text{kink}}(x) \sim \exp i\pi \int_{x_1}^{\infty} dz^1 \chi^*(z^1, x^0)\, \chi(z^1, x^0) . \tag{4.7}$$

A comparison between (4.6) and (4.7) immediately shows that whereas the coherent state is responsible for charge sectors with an additive quantum number, the successive application of the soliton operator (4.6) on the vacuum leading to inequivalent sectors, (4.7) creates a sector which is most conveniently labelled by a multiplicative quantum number (-1) since the successive application of two kink operators leads one to a state

$$|2 \text{ kink}\rangle \sim \exp i\, 2\pi \int^{\infty} dz' \chi^*(z')\, \chi(z')\, |0\rangle$$

which is equivalent to the vacuum sector.

5. A Glance at Higher Dimensions

The charge sectors of the since-Gordon theory are, as it was argued in section 2, a direct reflection of the existence of finite energy classical solutions with a different behavior at $x^1 = \pm\infty$ corresponding to a charge associated to the identically conserved current

$$j^\mu = -\frac{\beta}{2\pi}\, \varepsilon^{\mu\nu}\, \partial_\nu \phi \tag{5.1}$$

given by

$$Q = \int j^0(x)\, dx^1 = \frac{\beta}{2\pi}\left(\phi(\infty) - \phi(\infty)\right). \tag{5.2}$$

The simplest generalization of this feature to 4 dimensional space-time is given by the 't Hooft-Polyakov monopole [4].

In higher space-time dimensions very little has been done beyond the classical or semi-classical [2, 22, 36] approximation. We will briefly describe the classical features of the 't Hooft-Polyakov monopole which parallel the sine-Gordon theory.

One starts with an $SU(2)$ gauge theory coupled to an iso-triplet of Higgs [29] fields whose vacuum solution is

$$\langle\vec{\phi}\rangle = \phi_0\vec{\varepsilon}, \qquad \vec{A}_\mu = 0 \tag{5.3}$$

with ε a constant unit vector in isospin spcae. Finiteness of the energy requires that any solution should behave as the vacuum at spatial infinity

$$\underset{r\to\infty}{|\vec{\phi}|} \to \phi_0, \qquad \underset{r\to\infty}{D_\nu\vec{\phi}} \to O(r^{-2}), \tag{5.4}$$

where D_ν is the covariant derivative. From

$$\vec{F}^{\mu\nu} = \partial^\nu\vec{A}^\mu - \partial^\mu\vec{A}^\nu + e\vec{A}^\mu\Lambda\vec{A}^\nu \tag{5.5}$$

one can build a gauge invariant "electromagnetic" field

$$F^{\mu\nu} = \frac{\vec{\phi}\cdot\vec{F}^{\mu\nu}}{\phi_0} \tag{5.6}$$

which leads to an identically conserved magnetic current

$$k^\mu = \varepsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma}, \qquad \partial_\mu k^\mu = 0. \tag{5.7}$$

The simplest non trivial boundary condition of the field configuration satisfying (5.4) is given by

$$\vec{\phi} \to \phi_0\frac{\vec{r}}{r}, \qquad eA_i^a \to -\varepsilon_{iab}\frac{r_b}{r^2} \tag{5.8}$$

with i the spatial component and a the iso-spin index. A static solution satisfying (5.8) was proved to exist by 't Hooft and Polyakov [4]. The magnetic charge corresponding to (5.7) can be evaluated by Gauss law to be

$$\int k^0 d^3x = \int_{S\to\infty} \boldsymbol{B}\cdot d\boldsymbol{S} = \frac{4\pi}{e}. \tag{5.9}$$

The analogy between (5.2) and (5.9) is perfect. In the quantized version one therefore expects this theory to exhibit magnetic sectors corresponding to multiples of the fundamental magnetic charge (5.9) [1, 40].

This model was recently enriched by Hasenfratz and 't Hooft and Jackiw and Rebbi [37] who introduced besides the Higgs field an additional iso-spinor scalar field. The bound states of the magnetic monopole and the charged iso-spinor have half integer angular momentum [38] and should therefore correspond to fermions arising from a purely bosonic theory. One is therefore arriving at a 4-dimensional bosonization.

Gauge theories also provide one with a rather plausible mechanism for confinement in 4-dimensional space-time [7]. The main problem is to understand how a 4-dimensional field theory might be effectively reduced to a 2 dimensional one.

A simple example for such a reduction was provided by 'T HOOFT and KOGUT and SUSSKIND [7]. Considering electrostatics in a dielectric medium one has

$$H = \frac{1}{2} \int \frac{\boldsymbol{D} \cdot \boldsymbol{D} d^3x}{\varepsilon(\phi)} + H_{\text{medium}}.$$

Taking

$$H_{\text{medium}} = \frac{1}{2} \int (\nabla\phi)^2 + \mu^2\phi^2 \, d^3x$$

and

$$\varepsilon(\phi) \to 0$$

$$\phi \to 0$$

as a phenomenological description of infrared slavery the minimum of energy is obtained as a result of two competing tendencies. On the one hand the electrostatic energy (of a dipole) likes to spread as much as possible over the whole space. On the other hand the medium wants to be in its ground state $\phi = 0$ over as big a portion of space as possible. The net result is that the electric flux lines will be confined to a thin tube with the "Coulomb" potential between the pair growing linearly as in the two-dimensional case.
A similar confinement of flux lines arises naturally if one considers magnetic monopoles in a relativistic version of a superconductor due to the flux quantization condition of the latter [39].
The investigation of gauge-theories in a lattice by WILSON and KOGUT and SUSSKIND [7] also shows that there is a natural mechanism of flux quantization arising there.
Once an effective reduction of the 4-dimensional problem has been achieved one expects that the Schwinger model provides one with at least a qualitatively sound description for confinement.
Whether any of those new ideas will prove relevant for our understanding of high-energy physics remains as yet an open problem. They do teach us in any case that a nonlinear field theory has a much richer structure than one could suspect by doing standart perturbation theory.
After almost half a century of existence the main question about quantum field theory seems still to be: what does it really describe? and not yet: does it provide a good description of nature?

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